Observation of stochastic coherence in coupled map lattices

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Chaotic evolution of *structures* in a coupled map lattice driven by identical noise on each site is studied (a structure is a group of neighboring lattice sites for which values of the dynamical variable follow a certain predefined pattern). The number of structures is seen to follow a *power-law decay* with length of the structure for a given noise strength. An interesting phenomenon, which we call *stochastic coherence*, is reported in which a rise of *bell-shaped* type in abundance and lifetimes of these structures is observed for a range of noise-strength values. [S1063-651X(97)03803-8]

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The coupled map lattice (CML) model has been observed to exhibit diverse spatiotemporal patterns and structures [1], and serves as a simple model for experimental systems such as Rayleigh-Bénard convection, Taylor-Couette flow, Belousov-Zhabotinsky reaction, etc. [2]. A major interest is in understanding formation of structures, localized in both space and time, in turbulent fluid [3]. The role of fluctuations in onset, selection, and evolution of such patterns and structures has been studied in some detail [4,5]. In this paper we report a novel phenomenon observed in the dynamics of structures in a chaotically evolving one-dimensional CML driven by identical noise. By a *structure* we mean a group of neighboring sites whose variable values follow a certain prespecified spatial pattern. Distribution of these structures during evolution of the lattice shows that their numbers exhibit *power-law decay* with length of the structure for a given noise strength, with an exponent which is a function of noise strength. It is observed that the average length of these structures shows a bell-shaped curve with a characteristic peak, as a function of noise strength. Similar behavior is observed for average lifetime of these structures during their evolution, within the same range of noise values. We call this phenomenon *stochastic coherence*.

We consider a one-dimensional CML of the form

$$x_{t+1}(i)=(1-\varepsilon)F(x_t(i))+\frac{\varepsilon}{2}[F(x_t(i-1))+F(x_t(i+1))]+\eta_t, \tag{1}$$

where $x_t(i)$, $i=1,2,\ldots,L$ is the value of the variable located at site i at time t, η_t is the additive noise, ε is the (nearest neighbor) coupling strength, and L is the size of the lattice. Logistic function $F(x)=\mu x(1-x)$ is used as local dynamics governing nonlinear site evolution with nonlinearity parameter μ. We have used both open-boundary conditions $x_t(0)=x_t(1)$, $x_t(L+1)=x_t(L)$, and periodic-boundary conditions $x_t(L+i)=x_t(i)$, for our system. For noise η_t we have used a uniformly distributed random number bounded between $-W$ and $+W$, where W is defined as the *noise-strength* parameter.

We define a *structure* as a region of space in which the dynamical variables at sites within this region follow a predefined spatial pattern [6]. To study coherence in the system we choose a spatial pattern where the difference in the values of the variables of neighboring sites within the structure is less than a predefined small positive number, say δ, i.e., $|x_t(i)-x_t(i\pm 1)|\leqslant\delta$. We call δ the *structure parameter*. We look for such patterns, or coherent structures, to appear in the course of evolution of the model given by Eq. (1).

We now present our main observations. Values of μ, ε, and L are chosen so that the resultant dynamics of the system is chaotic. Coherent structures with length <3 (sites) and lifetime <2 (time steps) are disregarded in our observation. Values for W are chosen within the range [0,1]. Figure 1 shows a plot (on log-log scale) of distribution of number $n(l)$ vs length l of structures for different values of W, with $\mu=4$, $\varepsilon=0.6$, $\delta=0.0001$, and $L=10\,000$, and open-boundary conditions are used. The power-law nature of decay of $n(l)$ is clearly evident, and has a form

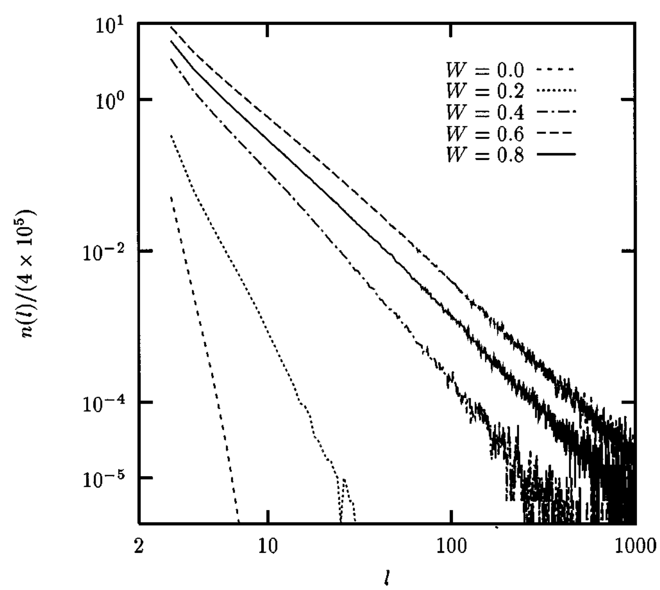

FIG. 1. Plot of variation of number $n(l)$ of structures with length l for a lattice with size $L=10\,000$, for different values of noise strength W as indicated. Parameters chosen are coupling strength parameter $\varepsilon=0.6$, structure parameter $\delta=0.0001$, and nonlinearity parameter $\mu=4$. Open-boundary conditions are used. Data are obtained for 100 000 iterates per initial condition and four initial conditions.

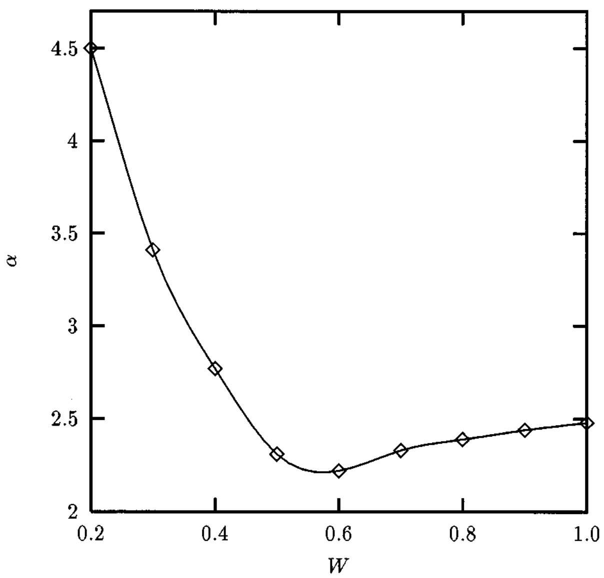

FIG. 2. Variation of exponent α [relation (2)] with noise strength W plotted for parameter values as in Fig. 1.

$$n(l)\propto l^{-\alpha}, \tag{2}$$

where α is the power-law exponent. This indicates that the system does not have any intrinsic length scale. It may be noted that in the absence of noise ($W=0$) the decay is manifestly exponential [6]. Exponent α (i.e., slope of the log-log plot in Fig. 1) is seen to depend on noise-strength W. This fact is corroborated in Fig. 2, which shows a plot of α vs W for the same parameter values as in Fig. 1. The exponent is exhibiting a clear minimum for values of W around 0.6. We define average length $\bar{l}$ of a structure as $\bar{l}=\Sigma l n(l)/\Sigma n(l)$. In Fig. 3 we plot variation of $\bar{l}$ with W for values of parameters as in Fig. 1. The plot exhibits a *bell-shaped* nature within a fairly narrow range of W around value 0.6. It may be noted that the minimum of α in Fig. 2 also occurs for W quite close to this value.

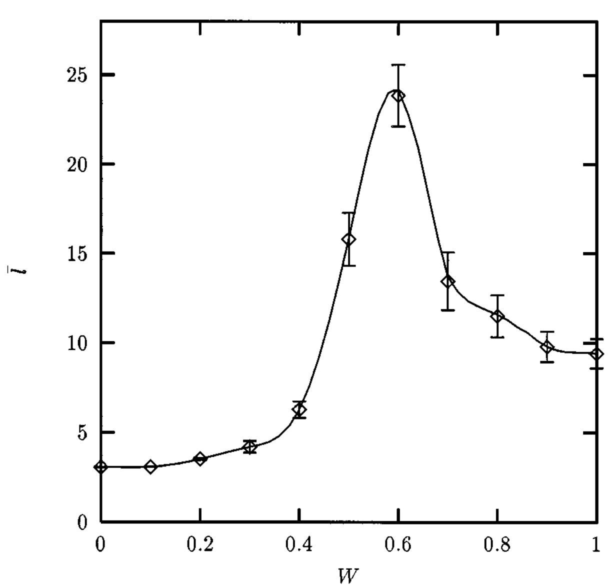

FIG. 3. Plot of variation of average length $\bar{l}$ of structure with noise strength W, with parameters as stated in Fig. 1.

We call the phenomenon observed above *stochastic coherence*. This is similar to the stochastic resonance phenomenon which shows a bell-shaped behavior of temporal response as a function of noise strength [7]. However, one may note that our system does not have any intrinsic length scale, whereas in stochastic resonance noise resonates with a given time scale; hence our use of the word coherence rather than resonance [8]. In stochastic resonance noise transfers energy to the system at a characteristic frequency, whereas in stochastic coherence noise *induces coherence* in the system. At this stage it is not clear to us how noise is inducing coherence in the system. However, absence of any length scale over the entire range of noise seems to indicate existence of weak self-organization in the system induced by noise.

In this context it should be noted that if a uniform-deviate random number is taken as site variable, then the probability of a site to belong to a structure of length l is $p_\delta(l)\approx l(2\delta)^{l-1}(1-2\delta)^2$, where δ is the structure parameter introduced earlier. Hence the number of such structures in a lattice of size L is given by $n_\delta(l)\approx L(2\delta)^{l-1}(1-2\delta)^2$. This shows an exponential decay of the number of such structures with length, which is to be contrasted with the power-law form (2) obtained for our lattice.

It is possible to show how noise helps to reduce instability of the structures, thereby increasing their abundance. Let us consider stability matrix M of a homogeneous state $\{\ldots,x_t,\ x_t,x_t,\ldots\}$ (this may be thought of as a large structure with $\delta=0$ for simplicity). At time $t+1$ the matrix takes the simple form $M_{t+1}=JF_t'$, where J stands for the familiar tridiagonal matrix (with diagonal elements $1-\varepsilon$ and off-diagonal elements $\varepsilon/2$ on either side) and $F_t'\equiv F'(x_t)(\equiv dF/dx)=\mu(1-2x_t)$. After two time steps, we get the stability matrix for three time steps as $S_3=M_{t+1}M_{t+2}M_{t+3}=J^3F_t'F_{t+1}'F_{t+2}'=J^3F_t'\mu^2[(1-2F_t)\{1-2\mu F_t(1-F_t)\}+6\mu\eta_t^2(1-2F_t)-2\eta_t\{1+\mu(1-6F_t+6F_t^2)\}-4\mu\eta_t^3-2\eta_{t+1}(1-2F_t)+4\eta_t\eta_{t+1}]$. For δ-correlated noise with uniform distribution and zero mean (which is the case here), after averaging the above expression over noise distribution one gets nonzero contribution due to noise only from the term quadratic in η:

$$\langle S_3\rangle=J^3(F_t')^2\mu[1-2\mu F_t(1-F_t)+6\mu\langle\eta_t^2\rangle], \tag{3}$$

where $\langle\ \rangle$ denotes averaging over noise. For our lattice with localized structures in the backdrop of spatiotemporal chaos we found the invariant density to be no longer symmetric, with larger weight for values of variable above 0.5. Averaging expression (3) over this density makes the term $1-2\mu F_t(1-F_t)$ negative. Adding positive contribution of noise to it results in reduction of eigenvalue of the matrix S_3 and hence consequent reduction in instability of the state.

To study evolutionary aspects of these coherent structures we obtained distribution of number $n(\tau)$ of structures vs their lifetimes τ for different W. This is shown in Fig. 4 (on a log-linear scale). It exhibits decrease of $n(\tau)$ with τ with a *stretched exponential* type of decay having a form

$$n(\tau)\propto\exp(-\text{const}\times\tau^{\beta}), \tag{4}$$

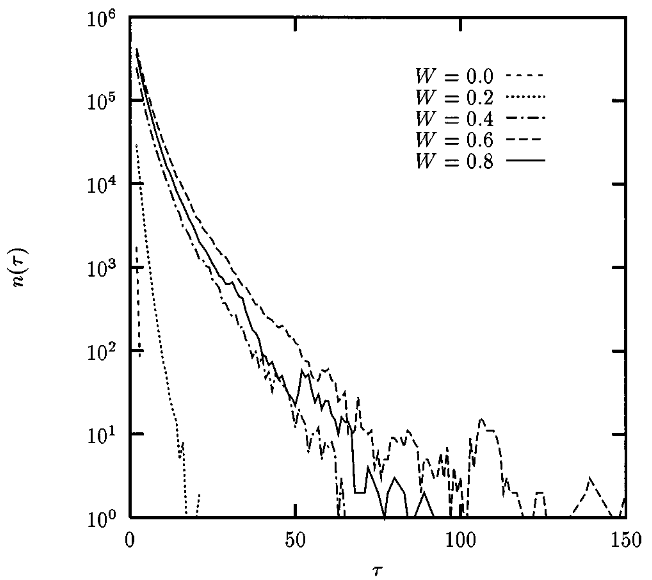

FIG. 4. Plot of variation of number $n(\tau)$ of structures with lifetime τ, with conditions as in Fig. 1.

where β depends on W. We define average lifetime $\overline{\tau}$ of a structure as $\overline{\tau}=\Sigma\tau n(\tau)/\Sigma n(\tau)$. In Fig. 5 we plot $\overline{\tau}$ vs W for parameter values as in Fig. 4. The graph also shows a bell-shaped feature with maximum for W around 0.6. This W value is close to that corresponding to the extrema in Figs. 2 and 3.

In order to ascertain the chaotic nature of system evolution we have calculated Lyapunov exponent spectra for our system. We find a number of Lyapunov exponents to be positive, implying that the underlying evolution is chaotic. Maximum Lyapunov exponent $\lambda_{\max}$ shows a minimum around noise strength 0.6 [9]. We have studied variation of λ spectrum with coupling strength ε. $\lambda_{\max}$ remains fairly constant for $0.2\leqslant\varepsilon\leqslant0.8$ for the entire range of W. Behavior

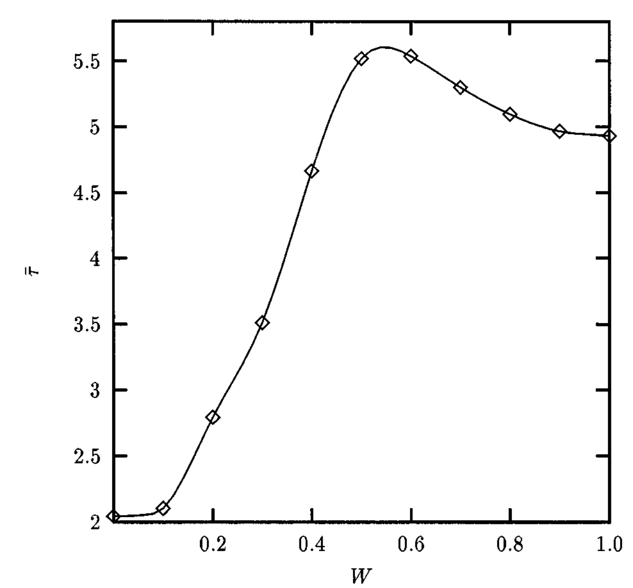

FIG. 5. Variation of average lifetime $\overline{\tau}$ of structures with noise strength W shown for parameters as stated earlier.

of $\overline{l}$ with ε is also investigated. It is found that $\overline{l}$ increases monotonically with ε for all W. This fact is quite contradictory to what is expected from $\lambda_{\max}$. This indicates that the Lyapunov exponent alone cannot be used for proper characterization of spatiotemporal features of the system. We have also obtained the power spectrum of the time series of the dynamical variable for a given site. The plot does not show any characteristic frequency and supports chaotic behavior. In addition, we have calculated the power spectrum of values $x_t(i)$, $i=1,\ldots,L$ at a given time. The nature of the spectrum confirms our earlier observation that the system does not have any intrinsic length scale. It may appear that the behavior of our system is similar to the spatiotemporal intermittency phenomenon [10]. However, our system does not show any regular burst-type feature (indicative of intermittency) in time; as already noted, the power spectrum of time series does not have any distinctive peak for the entire range of noise strength. Thus what we observe is development of appreciable coherence induced by noise in the system undergoing *spatially intermittent and temporally chaotic* evolution.

On quite a few occasions the entire lattice is seen to evolve as a single coherent structure to within the structure parameter δ. We have studied the lifetime of these coherent structures. Full lattice coherence appears for W above a value around 0.4. The interesting thing is that occasionally this state is seen to persist for fairly long durations (at times as long as 200 time steps or more), but eventually the coherence is seen to break up. This demonstrates that for our system the synchronized state is not a stable attractor.

However, in rare instances our system has been found to get into an apparently synchronized state after a very large time (larger than 10^7 steps). This is obtained because of the finite accuracy of computation which cannot distinguish an unstable synchronized state from a stable one [11]. As demonstrated in the following, for our system this type of behavior is an artifact resulting due to combination of finite size of the lattice and finite accuracy. Let us denote by $\overline{T}$ the average of time steps required for first occurrence of full lattice coherence, the average taken over different initial conditions. We have studied variation of $\overline{T}$ with structure parameter δ for a fixed lattice size L, which shows a definitive power law of the form $\overline{T}_L\propto\delta^{-\gamma}$, where γ is the power-law exponent. This indicates that the synchronized state, i.e., full lattice coherent structure with $\delta=0$, cannot be reached. We have also investigated $\overline{T}$ vs L variation for a fixed δ. We see a power-law behavior of form

$$\overline{T}_\delta\propto L^{\nu}, \tag{5}$$

where ν is the power-law exponent. Thus even full lattice coherence with finite δ cannot be achieved as $L\rightarrow\infty$.

We now show that the existence of full-lattice coherence with nonzero δ (distinct from synchronized state) in a finite lattice is essentially a consequence of power-law variation (2) of $n(l)$ with l. From relation (2) the probability of a site to belong to a structure of length l is $p_\delta(l)\propto l.l^{-\alpha}=l^{1-\alpha}$. Therefore the probability of a site to belong to a structure of length $\geqslant L$ is $P_0\equiv P_\delta(\geqslant L)\propto\Sigma_L^{\infty}l^{1-\alpha}\approx\int_L^{\infty}dl l^{1-\alpha}\propto L^{2-\alpha}$ (for $\alpha>2$, which is the case in the entire range of our observation as can be seen from Fig. 2). The probability that for the first time a site belongs to a structure of length $\geqslant L$ at time step T is then $P_\delta(T) \propto (1-P_0)^{T-1}P_0$. Therefore the average of T is given as $\overline{T_\delta}(\geqslant L) = \Sigma_{T=1}^{\infty} T P_\delta(T) \propto (P_0)^{-1}$, i.e.,

$$\overline{T_\delta}(\geqslant L) \propto L^{\alpha-2}. \tag{6}$$

This demonstrates that full-lattice coherence (with finite δ) will be observed in a finite lattice. Figure 2 tells us that $\alpha-2 \approx 0.22$ for $W=0.6$, whereas we obtain ν [relation (5)] ≈ 0.3. We believe the discrepancy to be due to boundary corrections, since Eq. (6) is obtained for an infinite lattice whereas Eq. (5) holds for finite lattices.

The above procedure was also carried out for several values of ε ranging from 0.1 to 0.9, as well as for nonlinearity parameter μ between 3.6 and 4. All the features remain essentially the same. We have also observed similar behavior with periodic-boundary conditions for the lattice.

In conclusion, we have reported a new phenomenon observed in a chaotically evolving one-dimensional CML driven by identical noise, which we termed *stochastic coherence*. It is observed that there is a phenomenal increase in abundance of coherent structures of all scales due to noise. By considering the stability matrix during three time steps, we have been able to show that noise can reduce instability of these structures. Distribution of these structures shows a power-law decay with length of the structure, with an exponent which shows a minimum at some intermediate noise strength. Average length as well as average lifetime of these structures exhibit characteristic maxima at a noise strength quite close to the previous value. This bell-shaped feature is similar to that of stochastic resonance, which is a temporal phenomenon. However, we emphasize that our system does not have any intrinsic length scale, whereas stochastic resonance is associated with a particular time scale. These observations demonstrate that noise can play a major role in formation as well as in evolutionary dynamics of structures in spatially extended systems.

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